This question paper contains **3** printed pages]

Y—56—2019

FACULTY OF ARTS/SCIENCE

B.A./B.Sc. (Fifth Semester) (Backlog) EXAMINATION

OCTOBER/NOVEMBER, 2019

(CBCS Pattern)

MATHEMATICS

Paper XII

(Metric Spaces)

(Thursday, 17-10-2019) **Time : 10.00 a.m. to 12.00 noon**

Time—2 *Hours* *Maximum Marks*—40

N.B. :— (*i*) Attempt *All* questions.

(*ii*) Figures to the right indicate full marks.

1. Attempt any *four* of the following : 2 each

(*a*) Define a metric d on a set X.

(*b*) Define adherent point in a metric space.

(*c*) Define isometry function in a metric space.

(*d*) State Banach fixed point theorem.

(*e*) Define connected metric space.

(*f*) Define finite intersection property.

2. Attempt any *two* of the following : 4 each

(*a*) In any metric space (X, d), prove that the intersection of a finite number of open sets is open.

(*b*) Let A and B be any two subsets of a metric space (X, d), then prove that :

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

P.T.O.

(*c*) Let (X, d) be any metric space. Show that the function d, defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \ \forall \ x, y \in X$$

is a metric on X.

3. Attempt any *one* of the following : 8 each

(*a*) (*i*) Prove that every convergent sequence is a Cauchy sequence.

(*ii*) Show that the space C[0, 1] of all bounded continuous real-valued functions defined on the closed interval [0, 1] with the metric d given by

$$d(f, g) = \max_{0 \le x \le 1} |f(x) - g(x)|$$

is a complete metric space.

(*b*) (*i*) Prove that the image of a Cauchy sequence under a uniformly continuous function is again a Cauchy sequence.

(*ii*) Let (X, d_1) and (Y, d_2) be metric spaces. Show that $f : X \to Y$ is continuous if and only if $f(\bar{A}) \subseteq \overline{f(A)}$, for every $A \subseteq X$.

4. Attempt any *two* of the following : 4 each

(*a*) Prove that every closed subset of a compact metric space is compact.

(*b*) Prove that continuous image of a connected set is connected.

(*c*) Let X be an infinite set with the discrete metric. Show that (X, d) is not compact.

5. Attempt any *one* of the following : 8 each

(*a*) Show that the set R^n of all ordered n tuples with the function d defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2},$$

for all $x = (x_1, x_2, \ldots\ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots\ldots, x_n) \in R^n$ is a metric space.

(*b*) Let (X, d_1) and (Y, d_2) be any two metric spaces and f is a continuous function from X into Y. Then prove that f is continuous at $a \in X$ if and only if, for every sequence $\{a_n\}$ converging to 'a', we have

$$\lim_{n \to \infty} f(a_n) = f(a).$$

(*c*) Let Y be a subset of a metric space (X, d), then prove the following statements are equivalent :

(*i*) Y is connected

(*ii*) Y cannot be expressed as disjoint union of two non-empty closed sets in Y.