UNIT:1 **WAVES**

1.1 PARTICLE VELOCITY AND WAVE VELOCITY:

Wave velocity : The distance covered by oscillatory disturbance per unit time in a medium is called wave Velocity.

Particle velocity: The velocity of vibrating particle about mean position.

is called particles Velocity

The standard equation of simple harmonic progressive wave is

$$y = a\,sin\frac{2\pi}{\lambda}(vt - x) \text{ ---- } (1)$$

Where

$\boldsymbol{a}$ is amplitude

λ is wavelength

V is wave velocity

Particle velocity is U= $\frac{dy}{dt}$

Differentiating equation (1) with respect to "t"

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \text{------ } (2)$$

Maximum value of particle velocity is

$$U_{max} = \frac{2\pi av}{\lambda} \text{ -----------------------}(3)$$

∴Maximum particle velocity= $\frac{2\pi a}{\lambda}$(wave velocity)

Particle Acceleration is $accn = \frac{d^2y}{dt^2}$

$$accn = -\frac{4\pi^2 av^2}{\lambda^2}\left[sin\frac{2\pi}{\lambda}(vt - x)\right] \text{----}(4)$$

Rewriting $accn = -\frac{4\pi^2 v^2}{\lambda^2}\left[asin\frac{2\pi}{\lambda}(vt - x)\right]$

$$accn = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) y$$

Acceleration is maximum when y=a

$\therefore accn_{max.} = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) a$ -- ---(5)

Negative sign indicates that the acceleration of the particle is directed towards its mean position.

Differentiating equation (1) with respect to "x"

$\frac{dy}{dx}$ = -$\frac{2\pi x}{\lambda}$ cos $\frac{2\pi}{\lambda}(vt - x)$ -----(6)

$\frac{dy}{dx}$ is slope of displacement curve

From equation (2) and (6) we have

$U = \frac{dy}{dt} = -v(\frac{dy}{dx})$ -------(7)

∴Particle velocity at any instant =wave velocity x slope of displacement curve

This is the relation between particle velocity and wave velocity

1.2 DIFFERENTIAL EQUATION OF WAVE MOTION:

Consider a SHPW. The displacement (y) of particle at a distance x from origin at instant of time (t) is

$$y = a\, sin\frac{2\pi}{\lambda}(vt - x) \text{ ---- (1)}$$

Where:

$\boldsymbol{a}$ is Amplitude

$\boldsymbol{\lambda}$ is Wavelength

$\boldsymbol{v}$ is Wave velocity

$\boldsymbol{t}$ is instantaneous time

$\boldsymbol{x}$ is position of particle from origin

Particle velocity is U= $\frac{dy}{dt}$

Differentiating equation (1) wrt "t"

$\frac{dy}{dt}$= $\frac{2\pi a v}{\lambda}$cos $\frac{2\pi}{\lambda}(vt - x)$-----(2)

Particle Acceleration is $accn = \frac{d^2y}{dt^2}$

Differentiating equation (2) wrt "t"

$\frac{d^2y}{dt^2} = -\frac{4\pi^2 a v^2}{\lambda^2}\left[sin\frac{2\pi}{\lambda}(vt-x)\right]$----(3)

Compression or strain is $\frac{dy}{dx}$

Differentiating equation (1) wrt "x"

$\frac{dy}{dx} = -\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$ -----(4)

Rate of change of Compression is $\frac{d^2y}{dx^2}$

Differentiating equation (4) wrt "x"

$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2}\left[sin\frac{2\pi}{\lambda}(vt-x)\right]$----(5)

From equation (2) and (4)

$\frac{dy}{dt} = -v\frac{dy}{dx}$ ----(6)

From equation (3) and (5)

$\boldsymbol{\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}}$ ---(7)

Equation (7) is a differential equation of wave motion

Any equation in this form always represent a wave motion

1.3 ENERGY OF A PROGRESSIVE WAVE:

In progressive wave energy is continuously transfer in the direction of propagation of waves. This energy is supplied by source. Energy transferred per second is also corresponds to energy possessed by the particles in a length "v" Energy of wave is partly kinetic and partly potential K.E. is due to velocity of vibrating particle. For a particle executing SHM , velocity is maximum at mean position and zero at extreme position . Therefore K.E. is maximum at mean position and zero at extreme position. Similarly, particle also possesses potential energy. P.E. is due to displacement of particle from mean position

P.E. is maximum at extreme position and minimum at mean position.

In longitudinal wave motion compression and rarefactions are produced

Energy distribution is not uniform over the wave

There is no transfer of medium in the direction of propagation of waves, but there is always transfer energy in the direction of propagation of waves

ANALYTICAL TREATMENT:

Consider a SHPW. The displacement (y) of particle at a distance x from origin at instant of time (t) is

$$y = a \sin\frac{2\pi}{\lambda}(vt - x) \text{ ---- (1)}$$

Where:

$\boldsymbol{a}$ is Amplitude

$\boldsymbol{\lambda}$ is Wavelength

$\boldsymbol{v}$ is Wave velocity

$\boldsymbol{t}$ is instantaneous time

$\boldsymbol{x}$ is position of particle from origin

Particle velocity is U= $\frac{dy}{dt}$

Differentiating equation (1) wrt "t"

$\frac{dy}{dt}$= $\frac{2\pi av}{\lambda}$cos $\frac{2\pi}{\lambda}(vt-x)$-----(2)

Particle Acceleration is $accn = \frac{d^2y}{dt^2}$

Differentiating equation (2) wrt "t"

$\frac{d^2y}{dt^2} = -\frac{4\pi^2av^2}{\lambda^2}\left[sin\frac{2\pi}{\lambda}(vt-x)\right]$----(3)

POTENTIAL ENERGY:

To move particle from its mean position to a distance y, work has to be done against acceleration.

Work done for displacement dy is

$$= F\,dy$$

If ρ density of medium

Work done per unit volume for dy

$$= \rho\left(\frac{4\pi^2av^2}{\lambda^2} sin\,\frac{2\pi}{\lambda}(vt-x)\right)dy$$

Total work done for displacement y is

$$= \int_0^y \rho\left(\frac{4\pi^2av^2}{\lambda^2} sin\,\frac{2\pi}{\lambda}(vt-x)\right)dy$$

But $y = a\,sin\frac{2\pi}{\lambda}(vt-x)$

$$\therefore Potential\ energy\ per\ unit\ volume$$

$$\left(\frac{4\pi^2\rho v^2}{\lambda^2}\right)\int_0^y ydy$$

$$= \frac{4\pi^2 \rho v^2}{2\lambda^2} y^2$$

$$= \frac{2\pi^2 \rho v^2}{\lambda^2} y^2$$

P. E per unit Volume =$\frac{2\pi^2 \rho v^2}{\lambda^2} a^2 sin^2 \left[\frac{2\pi}{\lambda}(vt - x)\right]$----(4)

K.E. per unit volume =$\frac{1}{2}\rho U^2$

$= \frac{1}{2}\rho \left[\frac{2\pi a v}{\lambda} cos \frac{2\pi}{\lambda}(vt - x)\right]^2$

$= \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} cos^2 \left[\frac{2\pi}{\lambda}(vt - x)\right]$----------------(5)

Adding eq. (4) and (5)

Total energy per unit volume

=$\frac{P.E.}{VOLUME} + \frac{K.E.}{VOLUME}$

=$\frac{2\pi^2 \rho v^2 a^2}{\lambda^2}\left[sin^2 \frac{2\pi}{\lambda}(vt - x) + cos^2 \frac{2\pi}{\lambda}(vt - x)\right]$

Total energy per unit volume $E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}$

$E = 2\pi^2 \rho n^2 a^2$ ------(6)

Av. K.E. per unit volume= $\pi^2 \rho n^2 a^2$

Av. P.E. per unit volume= $\pi^2 \rho n^2 a^2$

For unit area of cross section,

wave velocity $= v \quad Volume = 1 \times v = v$

Energy transferred per unit area per sec. =

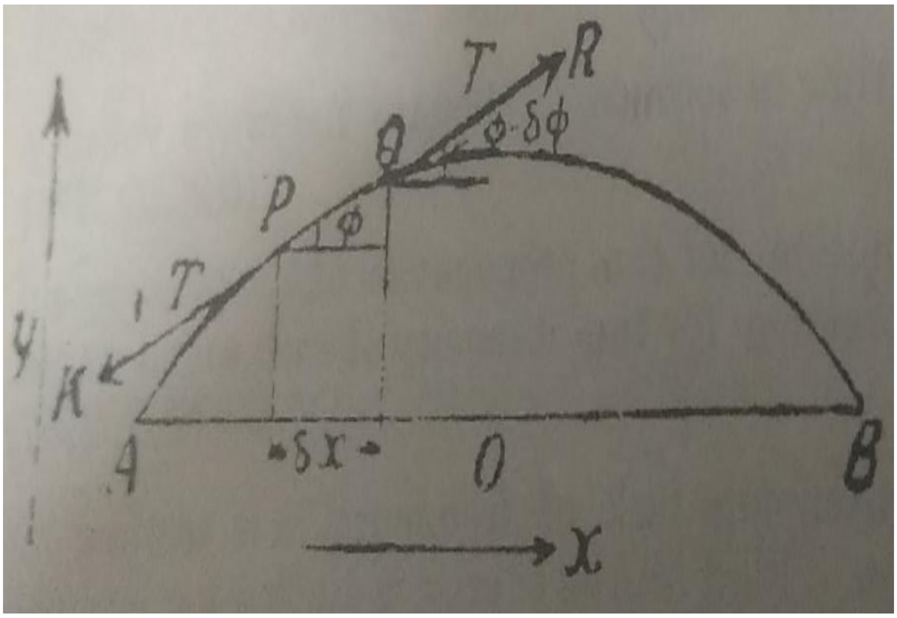
$$= \boldsymbol{E\, v = 2\pi^2 \rho v n^2 a^2}$$

1.4 EQUATION OF MOTION OF VIBRATING STRING:

The transverse vibrations of string constitutes one of the chief source of musical sound therefore the study of equation of motion of vibrating string is important.

Assumptions:

1. Length of string –greater than diameter
2 Perfectly uniform and flexible
3 Stretched between two fixed points
4 Tension in string should be large
5 Effect of gravitational force can be neglected.
6. Tension T in string should be constant everywhere



Let us consider that a string AB is stretched between two points with a tension T. It is plucked at the centre O so that transverse vibrations are produced in the string. The force tending to bring any element of string back to its potions

Of equilibrium is the components of tension T at right angles to AB. The displaced position of string is so small that tension T in the string remains almost constant. The magnitude of force acting on any element is proportional to the displacement and so motion is SHM.

Undisplaced position of string is represented along X axis and displaced position of string is represented along Y axis.

Consider a small element PQ of the plucked string of length δx acted upon by two tensions each equal to T along the tangents PK and QR.

Let ϕ and ϕ-δϕ angle of inclination of tangents to curve at P and Q respectively of element δx, ϕ itself being small.

Resolving tensions acting at ends of element into horizontal and vertical components. As horizontal components are equal and opposite so can be cancelled and the resultant of vertical components are in the direction of Y axis and is given by.

$\boldsymbol{R} = T sin$ϕ-T sin(f- δϕ)------(1)

$$R = T[sin\phi - sin(\phi - \delta\phi)]$$

$$R = T\ cos\ \phi\delta\phi = T\ \delta(sin\phi)$$

For small ϕ, sinϕ=tanϕ

From figure Tanϕ=$\frac{dy}{dx}$ =slope of displacement curve

$$\therefore \boldsymbol{R} = \boldsymbol{T}\,\delta\,(\frac{\boldsymbol{dy}}{\boldsymbol{dx}})$$

For distance δx

$$R = T\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]\delta x$$

$$R = T\frac{d^2y}{dx^2}\delta x\text{------(2)}$$

If m – mass per unit length of string,

Mass of element δx is= m δx

Acceleration in y direction= $\frac{d^2y}{dt^2}$

By Newtons second law of motion

Force=mass x acceleration

$$= m\delta x.\frac{d^2y}{dt^2} \quad \text{------(3)}$$

From equation (2) and (3)

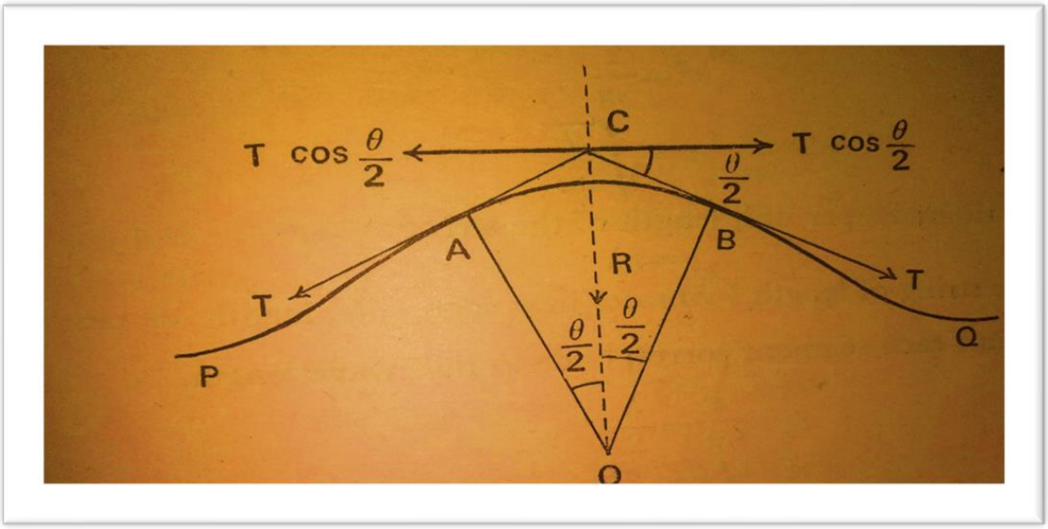
$$m\delta x.\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2}\delta x$$

$$\boldsymbol{\frac{d^2y}{dt^2} = \frac{T}{m}\frac{d^2y}{dx^2}}$$

This is differential equation of Vibrating string

1.5 VELOCITY OF TRANSVERSE WAVES ALONG STRETCHED STRING:

Consider a portion PABQ of a string in which transverse wave is traveling from left to right with a velocity v .



Let AB be small element of string.

O- is Centre of curvature of element AB

As curvature is small, θ small

Let T be Tension at A and B. The direction of these tensions are tangential to the element at A and B

Resolving tensions at A into two rectangular components

1) $T\,sin\frac{\theta}{2}$ Perpendicular to string in un displaced portion

2) $T\,cos\frac{\theta}{2}$ parallel to string in un displaced portion

Similarly

Resolving tensions at B into two rectangular components

1) $T\,sin\frac{\theta}{2}$ Perpendicular to string in un displaced portion

2) $T\,cos\frac{\theta}{2}$ parallel to string in un displaced portion

Parallel components cancel

Resultant perpendicular components along CO

Resultant tension along CO

$= 2T\,sin\frac{\theta}{2}$

as θ small $sin\frac{\theta}{2} = \frac{\theta}{2}$

Resultant tension= $2T\,\frac{\theta}{2} = T\theta$------(1)

For equilibrium position

Resultant tension provides necessary centripetal force

$= \frac{(m\delta x)v^2}{R}$ ------(2)

$= \frac{(m\delta x)v^2}{R} = T\theta$

But from fig Sinθ=$\frac{\delta x}{R}$ = θ=$\frac{\delta x}{R}$

$$\frac{(m\delta x)v^2}{R} = T\,\frac{\delta x}{R}$$

$$v^2 = \frac{T}{m}$$

$$v = \sqrt{\frac{T}{m}} \text{----(3)}$$

This is Velocity of transverse waves along stretched string.

1.6 FREQUENCY AND PERIOD OF VIBRATION OF A STRING.:

Let us consider a string fixed at both ends. If it is plucked at centre,

Transverse waves are set up in the string and travels towards fixed end.

At fixed end, debounce and reflect and start in reverse direction and goes into other end., gets reflect and arrives at original point after traversing twice the length of the string. If during this time point has executed one complete vibration.

During one period the displacement traversed by the wave is twice the length of the string.

Therefore $\lambda = 2l$ Where l is the length of the string.

$$n = \frac{v}{\lambda} = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

n is the lowest mode of vibration called fundamental mode of vibration and emitted note is called Fundamental

Period of vibration of fundamental mode

$$T = \frac{1}{n} = 2l\sqrt{\frac{m}{T}}$$

If D be diameter of wire

ρ - density of material of wire

Mass per unit length $m = \frac{Mass}{Length} = \frac{density \times Volume}{Length} = \frac{\rho\pi r^2 l}{l}$

$$\frac{\rho\pi r^2 l}{l} = \frac{\rho\pi D^2}{4}$$

Frequency of fundamental mode $n = \frac{1}{2l}\sqrt{\frac{4T}{\rho\pi D^2}} = \frac{1}{Dl}\sqrt{\frac{T}{\rho\pi}}$

Period $T = \frac{1}{n} = Dl\sqrt{\frac{\rho\pi}{T}}$

Using these formulae, one can calculate the fundamental frequency and period of vibration of stretched string